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THE PRODUCTION OF REGGE RECURRENCES

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Abstract: For resonance states lying on a given Regge trajectory, the two-body production mechanism as a function of the excitation of the recurrence state is discussed. A dual resonance model suggests general features for the Regge-Regge-particle coupling involved in such production. An application is made to the high energy production in $\pi N \to \pi\pi N$ of ρ, f_0 and g mesons with emphasis on the relative production cross sections, the relative t-dependences, the ratio of natural to unnatural parity exchange and the helicity dependence.

1. Introduction

There is no complete theory of two-body reactions at high energies. The salient features of the data, however, can be described in a t-channel complex angular momentum approach. The exchange of a Regge pole is found [1] to explain the energy dependence and phase of certain amplitudes (net helicity flip $n = 1$ in particular). The t-dependence of the Regge residue function can be evaluated from duality considerations. In particular the dual resonance model, B_4, gives a natural scale of $1/\alpha'$ to the energy s and so defines a residue $\beta(t)$ that should be essentially constant in t. The resulting t-dependence is indeed observed [1] experimentally for those amplitudes that have been found to be Regge-behaved. Other helicity amplitudes have a more complicated behaviour and in a complex angular momentum approach this implies the presence of Regge cuts.

Resonance states have been found to lie on exchange-degenerate Regge trajectories which are essentially linear in m^2. The highest lying such trajectory for a given set of quantum numbers (the parent or leading trajectory) is well established (for example $\rho - f_0 - g$; $\omega - A_2$; $K^* - K^{**}$ etc). The relative couplings (partial widths) of such Regge recurrence states to a given channel ($\pi\pi$ etc.) have been studied. For instance, a dual resonance B_4 model gives an $(\frac{1}{4}e)^J$ decrease of the partial widths for large J. The daughter states implied by such dual models are much more model dependent. They will be affected by any unitarization or any absorption of low partial waves needed to make the dual resonance model more physical.

At high energies, it becomes possible to produce Regge recurrence states in quasi two-body reactions, for example $\pi N \to \rho N$, $\pi N \to f_0 N$, $\pi N \to gN$ etc. At a given energy, from a combined study of the different states on the leading Regge trajectory†, one can discuss the J-dependence of (i) the resonance production cross section, (ii) the slope of the differential cross section in t and any eventual shrinking or anti-shrinking with J, (iii) the ratio of different exchanges (for example π to A_2 exchange in $\pi N \to \pi\pi N$) and the ratio of different helicity amplitudes, (iv) the relative absorption corrections to the Regge pole exchange.

The next section reviews models that allow a discussion of the relative production of Regge recurrences. The on-shell exchange coupling (or equivalently the decay matrix element) is first discussed. The off-shell or Regge exchange coupling relevant to the production process is then discussed in different models. The dual resonance model is found to give the most complete treatment and this is related to analyses [2] using the inclusive triple Regge limit and finite mass sum rules (duality for Regge-particle amplitudes). Appendix A contains relevant definitions and clarifications.

The third section, together with appendix B, represents the specific results for the particle-Regge exchange-Regge production vertex from dual resonance models. Compared to the finite mass sum rule approach, a similar decrease of the ratio of natural parity exchange to unnatural parity exchange with increasing excitation J is found, while, unlike that approach, no systematic anti-shrinking of the t-dependence with increasing mass is present in the dual model vertex. An analysis of the helicity structure of the Regge exchange coupling is made, and the problem of the unwanted crossing matrix zeroes in the s-channel helicity π-exchange amplitudes is resolved.

2. Regge recurrence production

As a prelude, the relative production cross section for a spin-J resonance of mass m in the process $a + b \to m + c$ by r-exchange is discussed. On the r-particle exchange pole, this can be related to the decay matrix element for $m \to a + r$. Thus, for π exchange from a π beam, the recurrence production cross section is related to the $\pi\pi$ partial width of the spin-J resonance, see also appendix A.

Historically, the partial widths of spin-J resonances were first estimated [3] from the centrifugal barrier suppression factors for a decay in a box of radius R

$$m_J \Gamma_J(m) \sim g_J q \,[qR\, h_J^{(1)}(qR)]^{-2} \,,$$

where $h_J^{(1)}$ is a spherical Hankel function of the first kind. For a linear Regge trajec-

† As well as the dependence of two-body reactions on $J = \alpha(m^2)$ of the produced parent state, the dependence on m^2 for fixed-J is also of interest. Thus, using vector dominance, one can relate the data on electroproduction ($m^2 < 0$) and photoproduction ($m^2 = 0$) of π mesons on nucleons to the data on $\pi N \to \pi\pi N$ with the dimeson system in a P-wave for a range of values of m^2 across the ρ meson width.

tory, $qR \sim \sqrt{J}$, and thus for large J, $\Gamma(J)$ decreases as $g_J J^{-J}$. With the dynamical assumption [4] of a constant g_J, this is a very rapid decrease of the partial width with J. The usefulness of the centrifugal barrier factor lies rather in describing the large q behaviour of the width for fixed-J which is independent of assumptions about g_J.

A more reliable estimate of the J-dependence of the coupling $\Gamma_{cd}(J)$ arises from considering the elastic scattering amplitude for c + d → c + d. For the imaginary part of the amplitude, duality relates the average direct channel resonance contribution to the Regge exchange amplitude. The t-dependence of the exchange amplitude then gives an estimate of the relative strength of different partial waves. For an amplitude with t-dependence $e^{\frac{1}{4}R^2 t}$, the contribution of the spin-J partial wave contains a factor

$$a(J) \sim e^{-J(J+1)/q^2R^2} .$$

Thus partial waves with $J \sim qR$ will be dominant while those with $J \gg qR$ will be suppressed. For a Regge pole exchange amplitude, $\frac{1}{4}R^2$ has the form $\alpha' \log(\alpha' s)$ and then $q^2R^2 \sim \alpha' s \log \alpha' s$. A direct channel resonance of partial width Γ_{cd} and total width Γ_T will contribute a bump to the imaginary part of the amplitude $a(J)$ of height Γ_{cd}/Γ_T and of extent in s (or m^2) of $m_J \Gamma_T$. Thus a_J will receive an average imaginary part of magitude $m_J \Gamma_{cd}$. Then such duality considerations allow an estimate of the coupling of a spin $J \sim \alpha' s = \alpha' m^2$ parent resonance to the channel cd:

$$m_J \, \Gamma_{cd}(J) \sim e^{-J/\log J} .$$

This is a much slower decrease with J than that found for the centrifugal barrier with constant g_J.

A more explicit example of such a dual estimation of the strength of the J^{th} partial wave from the t-dependence of the high energy amplitude is the B_4 dual resonance model itself. Using the $\pi\pi \to \pi\pi$ dual amplitude [5, 6] gives for the leading trajectory:

$$m_J \, \Gamma_{\pi\pi}(J) = \frac{q^2}{2J+1} \frac{q}{m} \frac{(2\alpha' q^2)^J}{\alpha'(J-1)!} \frac{1}{c_J} ,$$

where $c_J \sim 2^J$ for large J, and is defined in appendix B. When the relevant isospin factors are included, this expression gives a reasonable account [6] of the $\pi\pi$ partial widths: 0.85 Γ_ρ and 0.34 Γ_ρ are predicted for the f_0 and g respectively as against experimental values [7] of ~ 0.9 Γ_ρ and ~ 0.5 Γ_ρ. Comparison of the g and ρ partial widths provides the most interesting test, since this is insensitive to any exchange degeneracy breaking. For large J, and with linear trajectory $J = \alpha' m^2$, the above expression has a J-dependence

$$m_J\Gamma(J) \sim (\tfrac{1}{4}e)^J$$

Such a slow $(\frac{1}{4}e)^J$ exponential decrease is common to all dual model approaches.

In the production of Regge recurrence states at high energy, the coupling or Regge residue that enters is (see appendix A) $R^J_\lambda(t, m^2)$. When extrapolated to the particle exchange pole ($\alpha(t) = 0$ or 1 as appropriate). R is related to the decay matrix element as function of $J = \alpha(m^2)$ discussed above. Thus, the essentially new feature of interest in production is the interplay of the t, $J = \alpha(m^2)$, and λ dependence of the coupling.

The n-point dual resonance model allows [8] an explicit calculation of the coupling $R^J_\lambda(t, m^2)$ and the results are presented in appendix B and discussed in detail in the next section. Here, such a B_n calculation is compared with other approaches that have been suggested. The inclusive process a + b → anything + e, is related to the forward a + b + $\bar{e}$ → a + b + $\bar{e}$ amplitude. The exchange of a Regge trajectory $\alpha_r(t)$ coupled to b$\bar{e}$, then leads to the consideration of the forward Regge particle scattering amplitude r + a → r + a. Duality techniques applied to r + a → r + a relate the triple Regge amplitude at large m^2 to resonance contributions at small m^2. The triple Regge amplitude has a term $(s/m^2)^{2\alpha(t)}$ which correlates the t- and m^2-dependence, and this is conjectured [2] to be valid on average for m^2 in the resonance region. This gives rise to a production cross section for anything of mass m^2 which has a t-dependence antishrinking as $e^{-t2\alpha' \log m^2}$ with increasing m^2. The same factor $(s/m^2)^{2\alpha(t)}$ also comes† from taking the double Regge limit in the exclusive process a + b → c + d + e. Here duality techniques need to be applied [9] to the r + a → c + d amplitude for varying m^2.

The $(s/m^2)^{2\alpha(t)}$ factor also yields a faster fall off with m^2 when the exchanged trajectory $\alpha(t)$ is higher lying. Thus natural parity exchange ($\alpha(t) \sim 0.5 + t$) will become less important relative to unnatural parity exchange ($\alpha(t) \sim 0 + t$) as m^2 increases at fixed-s.

In practice for $\alpha(m^2)$ in the range 1 to 2, the leading trajectories should dominate the r + a → r + a and r + a → c + d processes, and such dual predictions will be relevant to parent resonance production. For higher m^2, however, just as for particle-particle scattering, the leading trajectory resonances will no longer dominate the amplitudes. Thus, there is no conflict with the result (sect. 3) from the dual resonance model that there is, in general, no antishrinking of the t-dependence with m^2 for the production of parent trajectory states.

Complementary to the complex angular momentum plane approach to the energy dependence, the dual absorptive model [10] or geometric model seeks to describe the momentum transfer dependence of two-body reactions. Thus the t-dependence of the production cross section is predicted in such a model, and its dependence on

† Reggeizing the two-body a + b → m + e production amplitude gives an expression $\beta(t)\, P_{\alpha(t)}(\cos\theta_t)$. For large s, at $t \neq 0$, $\cos\theta_t \to s(2q_{am}q_{be})^{-1}$; and for large mass m^2, $q_{am} \sim m^2(4t)^{-\frac{1}{2}}$. Thus, $P_{\alpha(t)}(\cos\theta_t)$ behaves as $(s/m^2)^{\alpha(t)}$. To have reasonable analytic behaviour, however, $\beta(t)$ must contain a factor $(q_{am}q_{be})^{\alpha(t)}$ and then the resultant two-body Regge exchange amplitude has no explicit kinematic dependence on m^2.

the external mass m of a produced resonance can be found. Indeed, if a universal radius of peripherality R_0 is supposed for all exchange reactions, then the t-dependence $J_\lambda(R_0\sqrt{-t})$ is independent of m^2. Remembering the motivation *via* duality with peripheral resonances at low energy, one might rather expect the average angular momentum $L_0 \sim qR$ to be independent of m^2. Then taking account of the dependence on m^2 of the final state momentum q_f, leads to a $J_\lambda(R_0\sqrt{-t}\,\sqrt{q}/\sqrt{q_f})$ behaviour. This shows a shrinking of the t-dependence with increasing m^2 since q_f decreases. For large s/m^2, however, this effect gives a t-dependence independent of m^2.

3. Regge recurrence excitation in the dual resonance model

Explicit calculations with a naturality conserving meson vertex and a naturality changing meson vertex are reproduced in appendix B. These vertices can be applied to natural parity meson production ($J^P = 0^+, 1^-, 2^+, 3^-$, etc.) from pseudo-scalar mesons by the exchange of natural or unnatural parity Regge trajectories. The applications will be most fruitful, if the amplitudes under consideration are Regge behaved in s-dependence and phase and have the t-dependence characteristic of dual resonance couplings.

For vector meson production on nucleons (PN → VN) the $n = 1$ natural parity isoscalar exchange amplitude ($\omega - f_0$) is indeed found [13] to have the Regge s-dependence: $(-t')^{\frac{1}{2}}(\alpha's)^{\alpha(t)}\,\Gamma(1-\alpha(t))\,\xi_\pm(t)$ where $\xi_\pm$ is the signature factor $(\mp 1 - e^{-i\pi\alpha(t)})$. The $\rho - A_2$ charge exchange $n = 1$ amplitude is relatively small and hard to isolate without polarization data. The $n = 1$ unnatural parity charge exchange (π) producing ρ with $\lambda = 0$ in the s-channel frame is also found [14, 15] to have the shrinking s-dependence of a Regge trajectory exchange. The t-dependence of this amplitude is found [16] to be $(\mu^2 - t)^{-1}\sqrt{-t}\,e^{bt}$ with $b = 4.4\ \mathrm{GeV}^{-2}$ at 17.2 GeV/c. This can be compared [1] with the Regge limit of a dual model expression $\beta(t)\,\Gamma(-\alpha(t))\,\xi_+(t)\,(\alpha's)^{\alpha(t)}$ where $\beta(t)$ is the product of the $\pi\rho$ and $N\bar{N}$ residues. In the range $0 < -t < 0.2\ \mathrm{GeV}^2$, this latter expression behaves approximately as $(\mu^2 - t)^{-1}\,\beta(t)\,e^{ct}$ where, at 17.2 GeV/c, $c = 3.8$ for $\alpha' = 0.9$ and $c = 4.3$ for $\alpha' = 1.0$. Comparing with the empirical values, the t-dependence of $\beta(t)$, thus defined with the duality scale of s of $1/\alpha'$, is almost constant apart from $\sqrt{-t}$ factors. This reggeized π-exchange gives a natural prediction for the exponential form factor that would have been needed for elementary π-exchange.

Bearing in mind these expectations of which amplitudes should be Regge behaved, some applications are presented of the dual resonance model couplings of Regge recurrences. Details are given in appendix B.

3.1. *The ratio of natural parity exchange to unnatural parity exchange*

The π exchange Regge couplings to recurrence states of spin J and helicity λ, $U_\lambda^J(t, m^2)$, has the structure of the decay matrix element of the state to $\pi\pi$, and an off-shell correction factor X^J_λ which, for the s-channel helicity frame, is the product of a low order polynomial in t and a $(-t)^{\frac{1}{2}|\lambda|}$ factor. For natural parity exchange, the Regge coupling $N^J_\lambda(t, m^2)$ has the structure of a decay matrix element, times a factor $\sqrt{-t}/p_a$ coming from the naturality change, and an off-shell correction factor. Because of this $\sqrt{-t}/p_a$ factor, $N^J_1(m^2)/U^J_0(m^2) \sim m^{-1}$ for increasing J, or $\alpha(m^2)$, at fixed-t. Explicitly evaluating the factors in appendix B, gives $N^J_1/U^J_0 \sim \sqrt{-t}$, $0.70\sqrt{-t}$ and $0.56\sqrt{-t}$ for $J = 1$, 2 and 3 production (ρ, f_0, g) respectively at $t \sim 0$.

This dependence is similar to the $(m^2)^{\alpha_U(0)-\alpha_N(0)}$ or m^{-1} dependence arising on average in the finite mass sum rule approach. Experimental evidence supporting this dependence has been given [2], in particular a comparison of $I_t = 0$ natural parity exchange and $I_t = 1$ unnatural parity exchange in $\pi N \to \rho N$ and $\pi N \to gN$. $\pi^- p \to \pi^- \pi^+ n$ data also show [15] a relative decrease of natural parity exchange with increasing $\pi\pi$ mass. The contribution of the π cut makes this more difficult to analyze quantitatively, however.

3.2. Slope dependence on m^2

The t dependences of the Regge vertices are given by the factors $X^J_\lambda(t, m^2)$ which are presented in appendix B. The t-dependence is different for different helicity amplitudes and also different for s- or t-channel frames. The t-dependence is not of exponential form, and is characterized by the linear term in t at small-t; $R^J_\lambda \sim (-t)^{\frac{1}{2}|\lambda|}(1 + b^J_\lambda t + \ldots)$. The slope of the t-dependence of the production amplitude as a function of excitation $J = \alpha(m^2)$ is characterized by $b(J)$. An antishrinking of the spin-J production cross section with t means a decrease of b with increasing m^2 at fixed energy s. For large-J and $\lambda_t = 0$, the π-exchange coupling does have an antishrinking behaviour of the slope b as approximately $b^J_{\lambda_t=0}(m^2) \sim -\frac{1}{2}\alpha' \log(m^2)$, similar to that found in the finite mass sum rule approaches from the $(s/m^2)^{\alpha(t)}$ factor. The same coupling in the s-channel frame ($\lambda_s = 0$), however, has a strongly shrinking behaviour $b^J_{\lambda_s=0}(m^2) \sim m^2$ coming from the crossing matrix. For the lowest spin states, one finds explicitly:

$b_\rho = 0$,	$b_{f_0} = 0.8$	and $b_g = 1.2$	for $\lambda_s = 0$ π exchange;
$b_\rho = 0$,	$b_{f_0} = 0$	and $b_g = 0.14$	for $\lambda_s = \pm 1$ natural parity exchange;
$b_\rho = 0$,	$b_{f_0} = -0.9$	and $b_g = -1.4$	for comparison from a $-\alpha' \log \alpha' m^2$

antishrinkage. These slope factors b^J represent the change in slope for different Regge recurrences at the meson vertex. An overall t-dependence coming from the Regge pole exchange factors and the baryon vertex have also to be added of course.

The $\lambda_s = 0$ π-exchange amplitude in $\pi N \to \rho N$ seems to be Regge behaved as discussed above. The comparison of this amplitude with those for f_0 and g production should be a particularly appropriate test of the predictions. Data indicate [15] that the s-channel helicity slope parameter is constant within errors from the ρ to f_0

region in $\pi N \to \pi\pi N$ at 17.2 GeV/*c*. The analysis assumes a common slope for all amplitudes at a given mass, however, although the ρ and f_0 $\lambda_s = 0$ amplitudes should dominate. A separation of these contributions and an extension to the g-meson region are needed to clarify experimentally the shrinkage or antishrinkage of the slope with J.

3.3. Crossing matrix zeroes

Consider π-exchange producing natural parity mesons. On the π-exchange pole, the only coupling in the t-channel helicity frame is $R^J_{\lambda_t=0}(m^2)$. Assuming that this is the only helicity coupling for all t, would then give an s-channel helicity structure $R^J_{\lambda_s}(t, m^2) = d^J_{\lambda_s 0}(\cos\chi(t))\, R^J_{\lambda_t=0}(t, m^2)$. Thus the zeroes of the d^J function in t would be present in the s-channel helicity couplings. For $J = 1$ production this yields $R^{J=1}_{\lambda_s=0}(t, m^2) \sim \cos\chi(t) \sim (t + m^2 - \mu^2)$ which has a zero in the physical region at $t = -m^2 + \mu^2 \sim -0.6$. At high energies, the data show [16] no sign of such a zero. Such zeroes can only be removed by introducing non-zero t-channel couplings to $\lambda_t \neq 0$. The full dual resonance model vertex, indeed, contains such reggeized π-couplings to $\lambda_t \neq 0$ amplitudes (vanishing at $\alpha(t) = 0$ of course). Then crossing to the s-channel helicity frame, the combination of contributions from different t-channel helicities no longer has the crossing matrix zeroes. Appendix B establishes this explicitly. For $J = 1$ production, for example, $R^{J=1}_{\lambda_s=0}(t, m^2)$ is a constant and the unwanted zero is removed naturally.

It is these additional helicity couplings of the Regge exchange that are important. They emerge naturally from the dual model structure and indicate that the s-channel helicity amplitudes have a simpler structure in t compared to the t-channel helicity amplitudes. Thus the dual model Regge vertex structure justifies the assumption of simple s-channel helicity couplings that have been made empirically. For instance, the surprising constancy [16] in t of the ratio γ_s of S- to P-wave $\pi\pi$ production in $\lambda_s = 0$ is naturally explained; this is related to the above discussion of the filling in of the crossing matrix zeroes. A model approach to π-exchange [17] takes the s-channel helicity amplitudes obtained by crossing the t-channel Born term and then arbitrarily replaces t by μ^2 in all factors except for the essential $(-t)^{\frac{1}{2}n}/(\mu^2 - t)$ dependence. This has the feature found naturally in the dual vertices of removing the t-structure coming from the crossing matrix, but also goes further since it introduces an absorption correction or cut in the $n = 0$ amplitude which has flip at both Regge vertices.

The additional Regge couplings also play a role in making the $\lambda_t = 2$ production amplitudes significant for t-values of the order of 1 GeV^2. Thus, taking as an example A_2 or K^{**} production by natural parity exchange, the helicity $\lambda_t = 2$ or $\lambda_s = 2$ contribution can be estimated from the formulae of appendix B and will be significant.

4. Conclusion

The most reliable component of the dual resonance model should be the vertex couplings for mesons on leading trajectories. Problems of daughter states, fermions, and unitarity corrections are thereby avoided. The Regge exchange coupling to produce leading Regge trajectory recurrence states has been evaluated to give the following results.

(a) The production amplitudes for Regge recurrence states can be predicted from a knowledge of the $J = 1$ amplitude. The resonance production amplitude decreases as $(\frac{1}{4}\mathrm{e})^{\frac{1}{2}J}$ for large J.
with increasing J.

(c) The variation of t-dependence with helicity and J has been discussed. Certain s-channel helicity amplitudes show a shrinkage of t-dependence with increasing J.

(d) Additional Regge couplings have been found that fill in the crossing matrix zeroes and so yield s-channel helicity amplitudes with simple t-dependence.

These results are simplest to apply in practice to amplitudes that are Regge behaved. Experimental evidence supporting (b) and (d) has been presented. Many specific predictions are contained in the couplings evaluated in appendix B. A successful analysis of the particle-Regge-Regge coupling may then give information that can be used to tackle the intriguing problem of understanding why the absorption corrections (or Regge cut effects) in the $n = 0$ $\pi N \to \pi\pi N$ amplitude appear [18, 15] to decrease with increasing m^2 (or J) relative to the π-pole contribution.

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Appendix A. Definitions of production amplitudes

Consider the process $a + b \to m + e$ where m is a spin-J resonance of helicity λ which decays with invariant mass m into two spinless particles c and d. In the rest frame of m, the direction of p_c is described by spherical polar angles θ and ϕ in a frame with O_y normal to the $a + b \to m + e$ scattering plane and O_z either along p_a in the t-channel frame or along $-p_e$ in the s-channel frame. The helicity amplitude for the $a + b \to c + d + e$ process can thus be factorized

$$A^{\mu_b\mu_a}_{\mu_e}(s, t, m^2, \cos\theta, \phi) = \sum_{\lambda} A^{\mu_b\mu_a J}_{\mu_e}(s, t, m^2)$$

$$\times\ [m_J^2 - m^2 - i m_J \Gamma_T(m^2)]^{-1} M_J\, d^J_{\lambda 0}(\cos\theta)\, e^{i\lambda\phi} \qquad (A.1)$$

where M_J is the decay matrix element of m into c + d and is related to the partial width by

$$m_J\,\Gamma_{cd}(m^2) = \frac{1}{2J+1}\,\frac{q}{8\pi m}\,|M_J|^2 \tag{A.2}$$

The exchange of a Regge pole r of trajectory $\alpha(t)$, signature τ, and lowest particle state of spin j_0, then gives an a + b → m + e amplitude:

$$A^{\mu_b\mu_a J}_{\mu_e\lambda}(s,t,m^2) = R^J_{\mu_a\lambda}(t,m^2)\,[-\tau - e^{-i\pi\alpha(t)}]\,(\alpha' s)^{\alpha(t)}\,\alpha'\Gamma(j_0-\alpha(t))R_{\mu_b\mu_e}(t) \tag{A.3}$$

$R^J_{\mu_a\lambda}(t,m^2)$ is the required coupling of the exchanged reggeon r of momentum transfer t to the incoming particle a and the produced state of mass m, spin J and helicity λ.

Some relations between these amplitudes and the observable differential cross sections are

A.1. Inclusive cross section a + b → anything + e.

From the generalized optical theorem, this can be related to the discontinuity of the forward three-particle a + b + $\bar{e}$ scattering amplitude $A(s,t,m^2)$

$$\frac{d\sigma}{dt\,dm^2} = \frac{1}{128\pi^2 s q_i^2}\,\mathrm{disc}\,A(s,t,m^2)\,. \tag{A.4}$$

An average over helicity labels is implied. For s/m^2 large and m^2 also large, a triple Regge behaviour has the form

$$A(s,t,m^2) \sim \left(\frac{s}{m^2}\right)^{2\alpha_r(t)} (m^2)^{\alpha_0(0)}\,, \tag{A.5}$$

where α_0 is the exchange trajectory (pomeron or other) intercept in the reggeon (r) · particle (a) total cross section.

A.2. Exclusive cross section a + b → c + d + e.

$$\frac{d\sigma}{dt\,dm^2\,d\Omega} = \frac{1}{64\pi s q_i^2}\,\frac{1}{(2\pi)^3}\,\frac{q_{cd}}{4m}\,|A^{\mu_b\mu_a}_{\mu_e}(s,t,m^2,\cos\theta,\phi)|^2\,, \tag{A.6}$$

where an average over initial and sum over final helicities is implied. For s/m^2 large and m^2 also large, there exists a double Regge limit with the form

$$A \sim \left(\frac{s}{m^2}\right)^{\alpha_r(t_{be})} (m^2)^{\alpha(t_{ac})} . \tag{A.7}$$

A.3. Spin J production in a + b → c + d + e and resonance production.

The $Y_L^M(\theta, \phi)$ moments of the observable of eq. (A.6) can be used to try and extract the spin-J, helicity-λ component in the cd final state channel. Resonance states in this channel should have Breit-Wigner shapes in their m^2-dependence. Integrating over the resonance line shape in m^2, and multiplying by $\Gamma^J_{tot}(m^2)/\Gamma^J_{cd}(m^2)$ to correct for the branching ratio, then gives the resonance production cross section. Because of unitarity, this is also related to the factorized production amplitude defined in eq. (A.1), by

$$\rho^J_{\lambda\lambda} \frac{d\sigma}{dt} = \frac{1}{64\pi s q_i^2} |A^{\mu_b \mu_a J}_{\mu_e \lambda}(s, t, m_J^2)|^2 , \tag{A.8}$$

where a helicity average over μ_a and μ_b and sum over μ_e is implied.

As a further clarification, the information contained in the m^2- or J-dependent observables A.1 to A.3 is illustrated at the π-exchange pole, $t = \mu^2$, in $\pi N \to \pi\pi N$: A.1 is related to $\sigma^{tot}_{\pi\pi}(m^2)$; A.2 is related to $(d\sigma/dt)_{\pi\pi}(m^2, \cos\theta)$; A.3 is related to $m_J\, \Gamma^J_{\pi\pi}(m_J)$.

Appendix B. Explicit dual resonance model couplings

Particle – Regge pole exchange – Regge recurrence production coupling residues $R^J_\lambda(t, m^2)$ are evaluated from dual resonance models. Isospin and signature factors are neglected, since the dynamical dependence on the variables is under study. Triple meson vertices are considered which are naturality conserving (for example $\pi + \pi \to \rho$, f_0, g etc.) or naturality changing (for example $\pi + A_2 \to \rho$, f_0, g etc., or $\pi + \rho \to \omega$, A_2 etc.).

B.1. Naturality conserving vertex

From factorizing the dual n-point function B_n into two pieces on a leading pole m at $\alpha(m^2) = J$ in an internal subenergy, the following expression for the process a + b → m + e arises [11]

$$V^J_{\mu_1 \dots \mu_J} = g^2 \frac{(\sqrt{2\alpha'})^J}{\sqrt{J!}} \int du\, u^{-1-\alpha(s)} (1-u)^{-1-\alpha(t)} \prod_{i=1}^{J} (p^\mu_a u - p^\mu_e(1-\mu)). \tag{B.1}$$

This must be contracted with the polarization tensor $\epsilon^J_{\mu_1 \ldots \mu_J}(\lambda)$ to obtain the helicity amplitudes. In the rest frame of m, where $\boldsymbol{p}$ has spherical polar co-ordinates θ and ϕ,

$$\epsilon^J_{\mu_1 \ldots \mu_J} p^{\mu_1} p^{\mu_2} \ldots p^{\mu_J} = \frac{(p)^J}{\sqrt{C_J}} d^J_{\lambda 0}(\cos\theta)\, e^{i\lambda\phi} \tag{B.2}$$

where $C_J = (2J)!\, 2^{-J}/(J!)^2$. A further property of the polarization tensor is

$$\epsilon^J_{\mu_1 \ldots \mu_J} = \sum_{\lambda_1 \lambda_2} (J-r, r, \lambda_2, \lambda_1 \,|\, J, \lambda)\, \epsilon^r_{\mu_1 \ldots \mu_r}(\lambda_1)\, \epsilon^{J-r}_{\mu_{r+1} \ldots \mu_J}(\lambda_2)\,. \tag{B.3}$$

Then expanding the product in eq. (B.1) gives terms in $(p^\mu_a)^r(-p^\mu_e)^{J-r}$ and eqs. (B.2) and (B.3) can be used to simplify the expression. In the s-channel helicity frame, $-\boldsymbol{p}_e$ is along O_z and has magnitude $s/2m$ for large-s, while $\boldsymbol{p}_a$ has z-component $p_a \cos\chi$ and x-component $p_a \sin\chi$ where

$$\begin{aligned} p^2_a(t) &= \lambda(m^2, t, a^2)/4m^2\,, \\ p_a(t)\cos\chi(t) &= (m^2 + t - a^2)/2m\,, \\ p_a(t)\sin\chi(t) &= (-t)^{\frac{1}{2}}\,. \end{aligned} \tag{B.4}$$

For large-s, the Regge limit of eq. (B.1), gives the Regge residue, defined as in appendix A,

$$U^J_\lambda(t, m^2) = g \frac{(2\alpha')^{\frac{1}{2}J}}{\sqrt{\alpha' J!\, c_J}} p^J_a(v^2)\, X^J_\lambda(t, m^2)\,, \tag{B.5}$$

where $\alpha(t) = \alpha'(t - v^2)$ and in the s-channel helicity frame

$$X^J_\lambda(t, m^2) = p_a^{-J}(v^2) \sum_{r=0}^{J} \frac{(p_a(t))^r (2\alpha' m)^{r-J}}{(J-r)!} \frac{\Gamma(-\alpha(t) + J - r)}{\Gamma(-\alpha(t))}$$

$$\times \left(\frac{(J+\lambda)!\,(J-\lambda)!}{(r+\lambda)!\,(r-\lambda)!}\right)^{\frac{1}{2}} d^r_{\lambda 0}(\cos\chi(t))\,. \tag{B.6}$$

This latter expression reduces to the product of a factor $(-t)^{\frac{1}{2}|\lambda|}$ and a polynomial in t of order $J - |\lambda|$ or less.

For the t-channel frame, the result is the same as eq. (B.5) and (B.6) except for the interchange of r and $J - r$ in the first three factors in the numerator of eq. (B.6).

On shell at $\alpha(t) = 0$ the expressions simplify to

$$X^J_{\lambda_s} = d^J_{\lambda_s 0}(\cos\chi(v^2)) , \qquad X^J_{\lambda_t} = \delta_{\lambda_t 0} . \tag{B.7}$$

The first term in (B.5) is the decay matrix element for m → a + on shell exchange particle. The above B_n results are for a theory of scalar particles with lowest states at $\alpha(m^2) = 0$. In practice, for the naturality conserving vertex π + (π exchange) → ρ, f_0, g etc., the produced resonances have trajectories with $\alpha(m^2) = 1$ as lowest state. This can be incorporated into a B_5 model for $\pi + \pi \to \pi + \pi + \epsilon$ (where ϵ is a $J^P = 0^+$ state) in the same way as into the B_4 model [5, 6] for $\pi\pi \to \pi\pi$. The resulting B_5 expression agrees with the B_4 matrix element on shell, and gives the same result for the Regge vertex as eq. (B.5) except for the replacement of the $\sqrt{J!}$ factor in the denominator by $\sqrt{(J-1)!}$.

The t-dependence of the correction factor $X(t)$ can be expressed, for small-t, as

$$X^J_\lambda(t, m^2) = (1 + t\, b^J_\lambda(m^2) + O(t^2))\,(-t)^{\frac{1}{2}|\lambda|} . \tag{B.8}$$

Neglecting a^2 and v^2, gives a general result

$$b^J_{\lambda_s=0} = b^J_{\lambda_t=0} + \frac{J(J+1)}{m^2} ,$$

$$b^J_{\lambda_t=0} = \frac{-2J}{m^2} - \alpha' \sum_{r=2}^{J} \frac{J!}{(J-r)!}\,\frac{1}{r}\,(\alpha' m^2)^{-r} . \tag{B.9}$$

Similarly, the t-dependence of $\Sigma_\lambda |X^J_\lambda|^2$ is characterized at small-t by $2\, b^J_{\lambda_t=0}$.

Explicit off-shell correction factors for $a^2 = v^2 = \mu^2$ and $p_a = \frac{1}{2}m(1 - 4\mu^2/m^2)^{\frac{1}{2}}$ in the s-channel helicity frame are

$$p_a X^1_0 = \tfrac{1}{2}m , \qquad p_a X^1_1 = -(-\tfrac{1}{2}t)^{\frac{1}{2}} ,$$

$$p_a^2 X^2_0 = \tfrac{1}{4}\{(m^2 + 2\mu^2) + (t - \mu^2)(2 - 1/\alpha' m^2)\} ,$$

$$p_a^2 X^2_1 = (-\tfrac{3}{8}t)^{\frac{1}{2}t}\, m , \qquad p_a^2 X^2_2 = -\tfrac{1}{4}\sqrt{6}\,t ,$$

$$p_a^3 X^3_0 = \frac{1}{8m}\{(m^4 + 6m^2\mu^2) + (t - \mu^2)(6m^2 - \frac{3}{\alpha'} - \frac{2}{\alpha' m^2 \alpha'})\} ,$$

$$p_a^3 X^3_1 = -\tfrac{1}{4}\sqrt{-3t}\,\{(m^2 + \mu^2) + (t - \mu^2)(1 - 1/\alpha' m^2)\} ,$$

$$p_a^3 X^3_2 = -\tfrac{1}{8}t\, m\sqrt{30} , \qquad p_a^3 X^3_3 = -\tfrac{1}{4}\sqrt{5}\,(-t)^{\frac{3}{2}} . \tag{B.10}$$

For the t-channel frame, similarly

$$p_a(t)X_0^1 = p_a(\mu^2)\left(1+\frac{3(\mu^2-t)}{m^2-4\mu^2}\right), \qquad p_a(t)X_1^1 = \frac{\sqrt{-t}\,(t-\mu^2)}{\sqrt{2}\sqrt{m^2-4\mu^2}\,m} \ . \tag{B.11}$$

B.2. Naturality changing vertex

From the B_5 dual resonance model for five pseudo-scalar mesons [12], one can extract the vertex for producing natural parity meson Regge recurrences by the exchange of a natural parity Regge trajectory $\alpha(t) - 1 = \alpha'(t - v^2)$. Extracting the leading trajectory spin J-pole in the cd channel and taking the Regge limit $s \to \infty$, gives a resonance production coupling together with the decay amplitude. Factorizing off the decay matrix element and picking out in the s-channel helicity frame the coefficient of $d^J_{\lambda 0}(\cos\theta)e^{i\lambda\phi}$ (see eq. (A.1)) gives the production Regge residue

$$N^J_\lambda(t, m^2) = \bar{g}\,\frac{(2\alpha')^{\frac{1}{2}J}}{\sqrt{(J-1)!}}\left(\frac{J+1}{\alpha' c_J J}\right)^{\frac{1}{2}} p^J_a(v^2)\,\frac{\sqrt{-t}}{p_a(v^2)}\,X^J_\lambda(t, m^2)\,,$$

$$X^J_\lambda(t, m^2) = p_a^{1-J}(v^2)\sum_{r=1}^{J}\frac{(p_a(t))^{r-1}(2\alpha' m)^{r-J}}{(J-r)!}$$

$$\times\,\frac{\Gamma(-\alpha(t)+J-r+1)}{\Gamma(-\alpha(t)+1)}\left(\frac{r(r+1)(J+\lambda)!\,(J-\lambda)!}{J(J+1)(r+\lambda)!\,(r-\lambda)!}\right)^{\frac{1}{2}}$$

$$\times\,\{d^r_{1\lambda}(\cos\chi(t)) + d^r_{-1\lambda}(\cos\chi(t))\}\ , \tag{B.12}$$

which is again a polynomial in t of order $J - |\lambda|$ or less together with a factor $(-t)^{\frac{1}{2}|\lambda|}$.

For on-shell exchange at $t = v^2$

$$X^J_\lambda(v^2, m^2) = d^J_{1\lambda}(\cos\alpha(v^2)) + d^J_{-1\lambda}(\cos\alpha(v^2))\,, \tag{B.13}$$

and in the t-channel frame

$$X^J_{\lambda_t}(v^2, m^2) = \delta_{\lambda_t \pm 1}\ . \tag{B.14}$$

Specific forms of the off-shell correction factor in the s-channel helicity frame, where $p_a = p_a(v^2)$ and $a = \mu$, are

$$X_0^J = 0\,,$$

$$X_1^1 = 1\,, \qquad p_{\mathrm{a}} X_1^2 = \frac{m^2 + v^2 - \mu^2}{2m}\,, \qquad p_{\mathrm{a}} X_2^2 = \sqrt{-t}\,,$$

$$p_{\mathrm{a}}^2 X_1^3 = \frac{1}{4m^2}\,[(m^2 + v^2 - \mu^2)^2 + m^2 v^2 + (t - v^2)(m^2 - 1/\alpha')]\,,$$

$$p_{\mathrm{a}}^2 X_2^3 = \sqrt{\tfrac{5}{2}}\,\frac{\sqrt{-t}}{2m}(m^2 + v^2 - \mu^2)\,, \qquad p_{\mathrm{a}}^2 X_3^3 = -\tfrac{1}{4}\sqrt{15}\,t\,. \tag{B.16}$$

In the t-channel helicity frame

$$p_{\mathrm{a}} X_1^2 = p_{\mathrm{a}}(t) + \frac{(1 - \alpha(t))\cos\chi(t)}{2\alpha' m}\,, \qquad p_{\mathrm{a}} X_2^2 = \frac{1 - \alpha(t)}{2\alpha' m}\sin\chi(t)\,. \tag{B.17}$$

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